

Solving Fractional Definite Integrals of Two Types of Fractional Trigonometric Functions

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Abstract: In this paper, based on Jumarie type of Riemann-Liouville (R-L) fractional integral and a new multiplication of fractional analytic functions, we study fractional definite integrals of two types of fractional trigonometric functions. The solutions of the two types of fractional definite integrals can be obtained by using the fractional trigonometric functions formula. In fact, our results are generalizations of classical calculus results.

Keywords: Jumarie type of R-L fractional integral, new multiplication, fractional analytic functions, fractional definite integrals.

I. INTRODUCTION

In the second half of the 20th century, a considerable number of studies on fractional calculus were published in the engineering literature. In fact, fractional calculus has many applications in physics, mechanics, biology, electrical engineering, viscoelasticity, control theory, economics, and other fields [1-17]. There is no doubt that fractional calculus has become an exciting new mathematical method to solve diverse problems in mathematics, science, and engineering.

Until now, the rules of fractional derivative are not unique. Many authors have given the definition of fractional derivative. The commonly used definition is the Riemann-Liouville (R-L) definition. Other useful definitions include Caputo definition of fractional derivative, Grunwald Letnikov (G-L) fractional derivative, conformable fractional derivative, and Jumarie's modified R-L fractional derivative [18-22]. Since Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with traditional calculus.

In this paper, based on Jumarie's modified R-L fractional integral and a new multiplication of fractional analytic functions, we study the following two types of α -fractional definite integrals:

$$\left({}_0I_{[\Gamma(\alpha+1)\frac{T\alpha}{2}]^{\frac{1}{\alpha}}}^{\alpha} \right) \left[[\sin_{\alpha}(2n\theta^{\alpha})] \otimes_{\alpha} [\sin_{\alpha}(\theta^{\alpha})]^{\otimes_{\alpha}(-1)} \right],$$

$$\left({}_0I_{[\Gamma(\alpha+1)\frac{T\alpha}{2}]^{\frac{1}{\alpha}}}^{\alpha} \right) \left[[\sin_{\alpha}((2n+1)\theta^{\alpha})] \otimes_{\alpha} [\sin_{\alpha}(\theta^{\alpha})]^{\otimes_{\alpha}(-1)} \right],$$

where $0 < \alpha \leq 1$, $\Gamma(\)$ is the gamma function, and n is any non-negative integer. The solutions of the two types of fractional definite integrals can be obtained by using the fractional trigonometric functions formula. Moreover, our results are generalizations of traditional calculus results.

II. PRELIMINARIES

At first, we introduce the fractional calculus used in this paper.

Definition 2.1 ([23]): Let $0 < \alpha \leq 1$, and θ_0 be a real number. The Jumarie's modified Riemann-Liouville (R-L) α -fractional derivative is defined by

$$({}_{\theta_0}D_{\theta}^{\alpha})[f(\theta)] = \frac{1}{\Gamma(1-\alpha)} \frac{d}{d\theta} \int_{\theta_0}^{\theta} \frac{f(t)-f(\theta_0)}{(\theta-t)^{\alpha}} dt, \tag{1}$$

and the Jumarie type of Riemann-Liouville α -fractional integral is defined by

$$({}_{\theta_0}I_{\theta}^{\alpha})[f(\theta)] = \frac{1}{\Gamma(\alpha)} \int_{\theta_0}^{\theta} \frac{f(t)}{(\theta-t)^{1-\alpha}} dt, \tag{2}$$

where $\Gamma(\)$ is the gamma function.

In the following, some properties of Jumarie type of R-L fractional derivative are introduced.

Proposition 2.2 ([24]): If $\alpha, \beta, \theta_0, c$ are real numbers and $\beta \geq \alpha > 0$, then

$$({}_{\theta_0}D_{\theta}^{\alpha})[(\theta - \theta_0)^{\beta}] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} (\theta - \theta_0)^{\beta-\alpha}, \tag{3}$$

and

$$({}_{\theta_0}D_{\theta}^{\alpha})[c] = 0. \tag{4}$$

Next, we introduce the definition of fractional analytic function.

Definition 2.3 ([25]): If θ, θ_0 , and a_k are real numbers for all k , $\theta_0 \in (a, b)$, and $0 < \alpha \leq 1$. If the function $f_{\alpha}: [a, b] \rightarrow R$ can be expressed as an α -fractional power series, i.e., $f_{\alpha}(\theta^{\alpha}) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)} (\theta - \theta_0)^{k\alpha}$ on some open interval containing θ_0 , then we say that $f_{\alpha}(\theta^{\alpha})$ is α -fractional analytic at θ_0 . Furthermore, if $f_{\alpha}: [a, b] \rightarrow R$ is continuous on closed interval $[a, b]$ and it is α -fractional analytic at every point in open interval (a, b) , then f_{α} is called an α -fractional analytic function on $[a, b]$.

In the following, we introduce a new multiplication of fractional analytic functions.

Definition 2.4 ([26]): Let $0 < \alpha \leq 1$, and θ_0 be a real number. If $f_{\alpha}(\theta^{\alpha})$ and $g_{\alpha}(\theta^{\alpha})$ are two α -fractional analytic functions defined on an interval containing θ_0 ,

$$f_{\alpha}(\theta^{\alpha}) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)} (\theta - \theta_0)^{n\alpha}, \tag{5}$$

$$g_{\alpha}(\theta^{\alpha}) = \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)} (\theta - \theta_0)^{n\alpha}. \tag{6}$$

Then we define

$$\begin{aligned} & f_{\alpha}(\theta^{\alpha}) \otimes_{\alpha} g_{\alpha}(\theta^{\alpha}) \\ &= \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)} (\theta - \theta_0)^{n\alpha} \otimes_{\alpha} \sum_{m=0}^{\infty} \frac{b_m}{\Gamma(m\alpha+1)} (\theta - \theta_0)^{m\alpha} \\ &= \sum_{n=0}^{\infty} \frac{1}{\Gamma(n\alpha+1)} \left(\sum_{m=0}^n \binom{n}{m} a_{n-m} b_m \right) (\theta - \theta_0)^{n\alpha}. \end{aligned} \tag{7}$$

Equivalently,

$$\begin{aligned} & f_{\alpha}(\theta^{\alpha}) \otimes_{\alpha} g_{\alpha}(\theta^{\alpha}) \\ &= \sum_{n=0}^{\infty} \frac{a_n}{n!} \left(\frac{1}{\Gamma(\alpha+1)} (\theta - \theta_0)^{\alpha} \right)^{\otimes_{\alpha} n} \otimes_{\alpha} \sum_{n=0}^{\infty} \frac{b_n}{n!} \left(\frac{1}{\Gamma(\alpha+1)} (\theta - \theta_0)^{\alpha} \right)^{\otimes_{\alpha} n} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{m=0}^n \binom{n}{m} a_{n-m} b_m \right) \left(\frac{1}{\Gamma(\alpha+1)} (\theta - \theta_0)^{\alpha} \right)^{\otimes_{\alpha} n}. \end{aligned} \tag{8}$$

Definition 2.5 ([27]): If $0 < \alpha \leq 1$, and θ is a real variable. The α -fractional exponential function is defined by

$$E_\alpha(\theta^\alpha) = \sum_{n=0}^{\infty} \frac{\theta^{n\alpha}}{\Gamma(n\alpha+1)} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{\Gamma(\alpha+1)} \theta^\alpha\right)^{\otimes_\alpha n} \tag{9}$$

On the other hand, the α -fractional cosine and sine function are defined as follows:

$$\cos_\alpha(\theta^\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^k \theta^{2n\alpha}}{\Gamma(2n\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{1}{\Gamma(\alpha+1)} \theta^\alpha\right)^{\otimes_\alpha 2n}, \tag{10}$$

and

$$\sin_\alpha(\theta^\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{1}{\Gamma(\alpha+1)} \theta^\alpha\right)^{\otimes_\alpha (2n+1)}. \tag{11}$$

Definition 2.6 ([28]): Let $0 < \alpha \leq 1$, and $f_\alpha(\theta^\alpha), g_\alpha(\theta^\alpha)$ be two α -fractional analytic functions. Then $(f_\alpha(\theta^\alpha))^{\otimes_\alpha n} = f_\alpha(\theta^\alpha) \otimes_\alpha \dots \otimes_\alpha f_\alpha(\theta^\alpha)$ is called the n th power of $f_\alpha(\theta^\alpha)$. On the other hand, if $f_\alpha(\theta^\alpha) \otimes_\alpha g_\alpha(\theta^\alpha) = 1$, then $g_\alpha(\theta^\alpha)$ is called the \otimes_α reciprocal of $f_\alpha(\theta^\alpha)$, and is denoted by $(f_\alpha(\theta^\alpha))^{\otimes_\alpha (-1)}$.

Definition 2.7 ([29]): The smallest positive real number T_α such that $E_\alpha(iT_\alpha) = 1$, is called the period of $E_\alpha(i\theta^\alpha)$.

III. MAIN RESULTS

In this section, we use the fractional trigonometric functions formula to solve two types of fractional definite integrals.

Theorem 3.1: If $0 < \alpha \leq 1$, and n is any non-negative integer, then

$$\left({}_0I^\alpha_{[\Gamma(\alpha+1)\frac{T_\alpha}{2}]^{\frac{1}{\alpha}}} \right) \left[[\sin_\alpha(2n\theta^\alpha)] \otimes_\alpha [\sin_\alpha(\theta^\alpha)]^{\otimes_\alpha (-1)} \right] = 0. \tag{12}$$

Proof Let $A_n = \left({}_0I^\alpha_{[\Gamma(\alpha+1)\frac{T_\alpha}{2}]^{\frac{1}{\alpha}}} \right) \left[[\sin_\alpha(2n\theta^\alpha)] \otimes_\alpha [\sin_\alpha(\theta^\alpha)]^{\otimes_\alpha (-1)} \right]$ for all non-negative integer n , then

$$\begin{aligned} & A_{n+1} - A_n \\ &= \left({}_0I^\alpha_{[\Gamma(\alpha+1)\frac{T_\alpha}{2}]^{\frac{1}{\alpha}}} \right) \left[[\sin_\alpha(2(n+1)\theta^\alpha)] \otimes_\alpha [\sin_\alpha(\theta^\alpha)]^{\otimes_\alpha (-1)} \right] - \\ & \left({}_0I^\alpha_{[\Gamma(\alpha+1)\frac{T_\alpha}{2}]^{\frac{1}{\alpha}}} \right) \left[[\sin_\alpha(2n\theta^\alpha)] \otimes_\alpha [\sin_\alpha(\theta^\alpha)]^{\otimes_\alpha (-1)} \right] \\ &= \left({}_0I^\alpha_{[\Gamma(\alpha+1)\frac{T_\alpha}{2}]^{\frac{1}{\alpha}}} \right) \left[[\sin_\alpha(2(n+1)\theta^\alpha) - \sin_\alpha(2n\theta^\alpha)] \otimes_\alpha [\sin_\alpha(\theta^\alpha)]^{\otimes_\alpha (-1)} \right] \\ &= \left({}_0I^\alpha_{[\Gamma(\alpha+1)\frac{T_\alpha}{2}]^{\frac{1}{\alpha}}} \right) \left[[2 \cdot \sin_\alpha(\theta^\alpha) \otimes_\alpha \cos_\alpha((2n+1)\theta^\alpha)] \otimes_\alpha [\sin_\alpha(\theta^\alpha)]^{\otimes_\alpha (-1)} \right] \\ &= 2 \cdot \left({}_0I^\alpha_{[\Gamma(\alpha+1)\frac{T_\alpha}{2}]^{\frac{1}{\alpha}}} \right) [\cos_\alpha((2n+1)\theta^\alpha)] \\ &= \frac{2}{2n+1} \cdot \left[\sin_\alpha\left((2n+1) \cdot \frac{T_\alpha}{2}\right) - \sin_\alpha(0) \right] \\ &= 0. \end{aligned}$$

Therefore, $A_n = A_0 = 0$ for any non-negative integer n .

Q.e.d.

Theorem 3.2: Let $0 < \alpha \leq 1$, and n be any non-negative integer, then

$$\left({}_0I^\alpha_{[\Gamma(\alpha+1), \frac{T_\alpha}{2}]^{\frac{1}{\alpha}}} \right) \left[[\sin_\alpha((2n+1)\theta^\alpha)] \otimes_\alpha [\sin_\alpha(\theta^\alpha)]^{\otimes_\alpha(-1)} \right] = \frac{T_\alpha}{2}. \tag{13}$$

Proof Let $B_n = \left({}_0I^\alpha_{[\Gamma(\alpha+1), \frac{T_\alpha}{2}]^{\frac{1}{\alpha}}} \right) \left[[\sin_\alpha((2n+1)\theta^\alpha)] \otimes_\alpha [\sin_\alpha(\theta^\alpha)]^{\otimes_\alpha(-1)} \right]$ for all non-negative integer n , then

$$\begin{aligned} & B_{n+1} - B_n \\ &= \left({}_0I^\alpha_{[\Gamma(\alpha+1), \frac{T_\alpha}{2}]^{\frac{1}{\alpha}}} \right) \left[[\sin_\alpha((2n+3)\theta^\alpha)] \otimes_\alpha [\sin_\alpha(\theta^\alpha)]^{\otimes_\alpha(-1)} \right] - \left({}_0I^\alpha_{[\Gamma(\alpha+1), \frac{T_\alpha}{2}]^{\frac{1}{\alpha}}} \right) \left[[\sin_\alpha((2n+1)\theta^\alpha)] \otimes_\alpha [\sin_\alpha(\theta^\alpha)]^{\otimes_\alpha(-1)} \right] \\ &= \left({}_0I^\alpha_{[\Gamma(\alpha+1), \frac{T_\alpha}{2}]^{\frac{1}{\alpha}}} \right) \left[[\sin_\alpha((2n+3)\theta^\alpha) - \sin_\alpha((2n+1)\theta^\alpha)] \otimes_\alpha [\sin_\alpha(\theta^\alpha)]^{\otimes_\alpha(-1)} \right] \\ &= \left({}_0I^\alpha_{[\Gamma(\alpha+1), \frac{T_\alpha}{2}]^{\frac{1}{\alpha}}} \right) \left[[2 \cdot \sin_\alpha(\theta^\alpha) \otimes_\alpha \cos_\alpha((2n+2)\theta^\alpha)] \otimes_\alpha [\sin_\alpha(\theta^\alpha)]^{\otimes_\alpha(-1)} \right] \\ &= 2 \cdot \left({}_0I^\alpha_{[\Gamma(\alpha+1), \frac{T_\alpha}{2}]^{\frac{1}{\alpha}}} \right) [\cos_\alpha((2n+2)\theta^\alpha)] \\ &= \frac{2}{2n+2} \cdot \left[\sin_\alpha \left((2n+2) \cdot \frac{T_\alpha}{2} \right) - \sin_\alpha(0) \right] \\ &= 0. \end{aligned}$$

Thus, $B_n = B_0 = \frac{T_\alpha}{2}$ for any non-negative integer n . Q.e.d.

IV. CONCLUSION

In this paper, based on Jumarie’s modified R-L fractional integral and a new multiplication of fractional analytic functions, we study fractional definite integrals of two types of fractional trigonometric functions. The solutions of the two types of fractional definite integrals can be obtained by using the fractional trigonometric functions formula. On the other hand, our results are generalizations of ordinary calculus results. In the future, we will continue to use Jumarie type of R-L fractional calculus and the new multiplication of fractional analytic functions to solve the problems in fractional differential equations and applied mathematics.

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